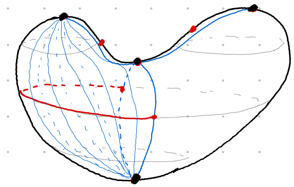
An Introduction to flow categories



1. Motivation and examples
2. The homotopy theory of flow categories
3. ∞-categories of flow categories

Based on the works:

- Porcelli - Smith (Parts 1. and 2.)
  "Bordism of flow modules and exact Lagrangians"

- Abouzaid - Blumberg (Part 3.)
  "Foundations of Floer homotopy 1: Flow categories"

Def A <u>graded flow category</u> X consists of

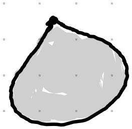
- $\mathrm{Ob}(X) \xrightarrow{|\cdot|} \mathbb{Z}$

- For $p, q \in \mathrm{Ob}(X)$, a compact manifold with ~~corners~~ faces $X(p,q)$ dim $= |p| - |q| - 1$

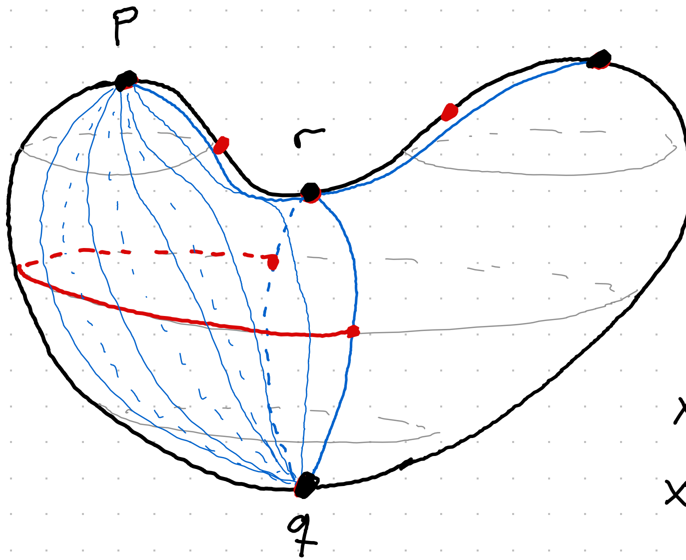
Such that:

- $\bigcup_r X(p,r)$ is compact

- $\partial X(p,q) = \bigcup_r X(p,r) \times X(r,q)$ $\supset X(p,r) \times X(r,r') \times X(r',q)$

<u>Ex</u>  is <u>not</u> mfld w/ faces. $\Delta^n, I^n, \ldots$ are

Ex morse theory $f : M \to \mathbb{R}$ morse



$|p| = 2$

$|r| = 1$

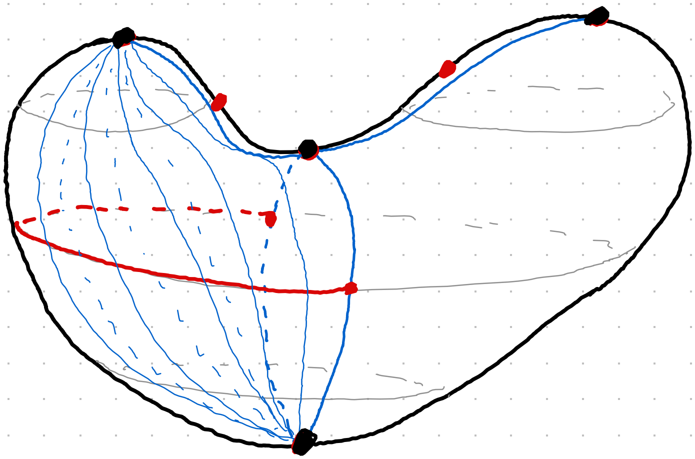
$|q| = 0$

$X(p,r) = *$

$X(r,q) = S^0$

$X(p,q) = I$

Ex morse theory



Ex $\emptyset$ and 1

Def For X a graded flow cat, let

$$C_*(X;\mathbb{Z}/2) = \bigoplus_{|p|=*} \mathbb{Z}\langle p\rangle \ , \quad \partial p = \sum_{|q|=|p|-1} \# X(p,q)\cdot q$$

Def $\Sigma^n X$ has

- $ob(\Sigma^n X) = Ob(X) \xrightarrow{|\cdot|+n} \mathbb{Z}$
- $(\Sigma^n X)(p,q) = X(p,q)$

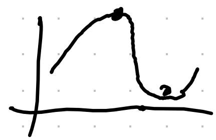
2. Homotopy theory of flow categories

Def A graded flow bimodule $F : X \to Y$ consists of:

- For every $p \in Ob(X)$, $q \in Ob(Y)$ a compact manifold w/ faces $F(p,q)$ $\dim = |p| - |q|$

such that:

- $\bigcup_r F(p,r)$ is compact
- $\partial F(p,q) = \begin{cases} \bigcup_r F(p,r) \times Y(r,q) \\ \bigcup_{r'} X(p,r') \times F(r',q) \end{cases}$



Ex A "generic" 1-parameter family of functions

Ex A bimodule $\Sigma^n 1 \to 1$ is a closed manifold

$$F(*,*)$$

$$(F \cup G)(p,q) = F(p,q) \cup G(p,q)$$

Def A <u>bordism</u> between F and G consists of:

- For every $p \in Ob(X)$, $q \in Ob(Y)$ a compact manifold w/ faces $H(p,q)$, $\dim = |p| - |q| + 1$

Such that:

- $\bigcup_r H(p,r)$ is compact

- $\partial H(p,q) = \begin{cases} \bigcup_r H(p,r) \times Y(r,q) \\ \bigcup_{r'} X(p,r') \times H(r',q) \\ F(p,q) \\ G(p,q) \end{cases}$

Ex "Generic" 2-parameter family of functions

Ex A bordism $B : M \sim N : \Sigma^n 1 \to 1$

<u>Slogan</u> Flow categories are 'chain complexes of manifolds up to cobordism

Flow category $\mathcal{X}$ $\partial \mathcal{X} = \mathcal{X} \times \mathcal{X}$	Chain complex C $0 = d_C^2$
Flow bimodule $F: \mathcal{X} \to \mathcal{Y}$ $\partial F = \begin{cases} \mathcal{X} \times F \\ F \times \mathcal{Y} \end{cases}$	Chain map $f: C \to D$ $0 = \begin{cases} d_C \circ f \\ f \circ d_D \end{cases}$
Bordism $B: F \sim G$ $\partial B = \begin{cases} \mathcal{X} \times B \\ B \times \mathcal{Y} \\ F \cup G \end{cases}$	Chain homotopy $h: f \sim g$ $0 = \begin{cases} d_C \circ h \\ h \circ d_D \\ f - g \end{cases}$

Def $[X,Y]_n := \{\text{bimodules } F\ \Sigma^n X \to Y\}/\text{bordism}$

Ex $[1,1]_n = \Omega_n^o$ acts by product on $[X,Y]_*$

M closed $\qquad F(p,q) \times M$

Theorem (Porcelli · Smith)

$\exists$ a (nonunital) category enriched in Ω_*^o, $h\mathrm{Flow}^o$

w/ $\mathrm{Ob}(h\mathrm{Flow}) = \{\text{(finite) graded flow cats}\}$ $\quad id_X(p,p) = *$

$\hom(X,Y) = [X,Y]_*$

$id : X \to X \quad id_X(p,q) = X \times I$

Rem we get a functor

$C_*(-;\mathbb{Z}/2) : h\mathrm{Flow}^o \to h\mathrm{Ch}(\mathbb{Z}/2)$

∞-Categories of flow categories

Theorem (Abouzaid - Blumberg)
$\exists$ a stable ∞-cat Flow^{o} of graded flow categories

How?

1) Construct monoidal categories

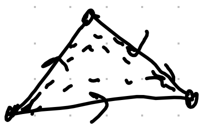
$$\mathrm{mfld}^{\partial}_{\diamond} \longrightarrow \mathrm{Poset}_{\diamond}$$

2) write down "master equation" for a flow-n-simplex

3) Define a flow n-simplex to be a category enriched in $\mathrm{mfld}^{\partial}_{\diamond}$ satisfying master eq.

4) Get semisimplicial set Flow^{o}

Δ^{n}



Master equation for 2-simplex:

$$\begin{tikzcd} & Y \arrow[dr, "G"] \arrow[d, phantom, "\Updownarrow"] & \\ X \arrow[ur, "F"] \arrow[rr, "H"'] & {} & Z \end{tikzcd}$$

$|p|-|q|+1$

$\forall\, p \in ob(X)$
$q \in ob(Z)$ $\quad B(p,q)$

$$\partial B(p,q) = \begin{cases} \bigcup_r X(p,r) \times B(r,q) \\ \bigcup_{r'} B(p,r) \times Y(r,q) \\ H(p,q) \\ \bigcup_r F(p,r) \times G(r,q) \end{cases}$$

$B \otimes_A B'$

B is a bordism "$F \otimes_Y G$" $\sim H$

<u>Slogan</u> Flow° is a morita cat of $\mathrm{Mfld}^\circ_\square$, localized at bordism.

Given $B: X \to Y$ $\qquad V_p$

Let $C(B)$ with objects $\Sigma\, ob(X) \amalg ob(Y)$ $\qquad V_p \oplus \mathbb{R}$

$$C(B)(p,q) = \begin{cases} X(p,q) & \text{if } p,q \in ob(X) \\ Y(p,q) & \text{if } p,q \in ob(Y) \\ B(p,q) & \text{if } p \in ob(X),\ q \in ob(Y) \\ \varnothing & \end{cases}$$

$Y \to C(B) \to \Sigma X$

<u>Prop</u> $C(B)$ is a cofiber of B

Prop Flow^{0} has inner horn fillers and "quasi-units"

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathrm{Flow}^0 \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad 0<i<n$$

Prop Flow^{0} is a presentable stable ∞-cat.

- $\varnothing$ is a 0-object
- Disjoint union gives $\coprod_\alpha X_\alpha$
- For $F: X \to Y$, let $C(F)$ have

$$\mathrm{Ob}(C(F)) = \mathrm{Ob}(\Sigma X) \cup \mathrm{Ob}(Y)$$

$$C(F)(p,q) = \begin{cases} X(p,q) & (|p|+1-|q|-1)-1 = |p|-|q|-1 = \dim X(p,q) \\ Y(p,q) & \\ F(p,q) & |p|+1-|q|-1 \\ \varnothing & |p|-|q| = \dim F(p,q) \end{cases}$$

- 1 generates

$\Sigma^n 1 \qquad X$

$\mathrm{End}(1)$

$$[1,1]_* \simeq \Omega^{\circ}_*$$

<u>Conjecture</u> $\mathrm{map}(1,1) \simeq MO$

<u>Cor</u> $\mathrm{map}(1,-) : \mathrm{Flow}^{O} \xrightarrow{\sim} MO\text{-Mod}$

<u>Prop</u> $\exists$ $\mathrm{Flow}^{Fr} \simeq Sp$

$$\Omega^{Fr}_* = \pi_* \mathbb{S}$$

$$\mathrm{Flow}^{Fr}(1,1) \simeq \mathbb{S}$$